ON THE CROUZEIX RATIO FOR $N \times N$ MATRICES

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ABSTRACT. The Crouzeix ratio $\psi(A)$ of an $N \times N$ complex matrix A is the supremum of $\|p(A)\|$ taken over all polynomials p such that $|p| \le 1$ on the numerical range of A. It is known that $\psi(A) \le 1 + \sqrt{2}$, and it is conjectured that $\psi(A) \le 2$. In this note, we show that $\psi(A) \le C_N$, where C_N is a constant depending only on N and satisfying $C_N < 1+\sqrt{2}$. The proof is based on a study of the continuity properties of the map $A \mapsto \psi(A)$.

1. Introduction

Let H be a complex Hilbert space, and let $\mathcal{B}(H)$ be the algebra of bounded linear operators on H, equipped with the operator norm. Given $T \in \mathcal{B}(H)$, we write $\sigma(T)$ for the spectrum of T, and $W(T)$ for the *numerical range* of T, namely the set

$$W(T) := \big\{\langle Tx, x\rangle : x \in H, \|x\| = 1\big\}.$$

It is well known that $W(T)$ is a bounded convex subset of $\mathbb{C}$ whose closure contains $\sigma(T)$. If further $\dim H < \infty$, then $W(T)$ is also compact.

The central object of study in this note is the *Crouzeix ratio* of $T \in \mathcal{B}(H)$, defined by

$$\psi(T) := \sup\Big\{\|p(T)\| : p \text{ is a polynomial}, |p| \le 1 \text{ on } W(T)\Big\}. \tag{1}$$

It was first studied by Crouzeix in [5]. The terminology 'Crouzeix ratio' is taken from [7] and [14]. We always have $\psi(T) \ge 1$ (consider $p \equiv 1$), with equality if T is a normal operator. It is also easy to see that $\psi(U^*TU) = \psi(T)$ for all unitary operators U, and that $\psi(\alpha T + \beta I) = \psi(T)$ for all $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$.

It is not obvious, *a priori*, that the Crouzeix ratio is always finite. That this is indeed the case was first proved by Delyon and Delyon in [9]. This

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is sometimes expressed by saying that $W(T)$ is a $\psi(T)$-*spectral set* for T. As a consequence of their result, the homomorphism $p \mapsto p(T)$ extends by continuity to a homomorphism $f \mapsto f(T)$ defined for all $f \in A(W)$, where $W = \overline{W(T)}$, and $A(W)$ is the uniform algebra of all continuous functions on W that are holomorphic on W°, the interior of W. The extended map, often called the functional calculus for T, satisfies

$$\|f(T)\| \le \psi(T) \sup_W |f| \quad (f \in A(W)). \tag{2}$$

The same article also contains a result [9, Theorem 3] that implies a quantitative bound for $\psi(T)$, namely

$$\psi(T) \le \Big(\frac{2\pi \operatorname{diam}(W)^2}{\operatorname{area}(W)}\Big)^3 + 3.$$

However, this does not yield a universal numerical bound for $\psi(T)$. The first such bound was obtained by Crouzeix, who showed in [6, Theorems 1 and 2] that we always have

$$\psi(T) \le 11{\cdot}08.$$

Some years later, Crouzeix and Palencia [8, Theorem 3.1] improved this estimate to

$$\psi(T) \le 1 + \sqrt{2}, \tag{3}$$

at the same time greatly simplifying the proof. Recently, this bound was further improved to

$$\psi(T) \le 1 + \sqrt{1 + a(W)}, \tag{4}$$

where, once again $W = \overline{W(T)}$, and where $a(W)$ is the so-called analytic configuration constant of W, which is a number depending only on W and satisfying $0 \le a(W) < 1$ (see [13, Theorem 2 and Proposition 26]). In particular, we always have

$$\psi(T) < 1 + \sqrt{2}, \tag{5}$$

so the Crouzeix–Palencia bound in (3) is never attained.

To continue the discussion, it will be convenient to introduce a further piece of notation. For each $N \ge 1$, we write $M_N(\mathbb{C})$ for the algebra of complex $N \times N$ matrices, and define

$$C_N := \sup\{\psi(A) : A \in M_N(\mathbb{C})\}. \tag{6}$$

It is easy to see that $C_1 = 1$ and that $C_N \le C_{N+1}$ for all $N \ge 1$. Clearly, from (3), we have $C_N \le 1 + \sqrt{2}$ for all N, so C_N converges to a limit C, say, where $C \le 1 + \sqrt{2}$. An approximation argument (see e.g. [6, Theorem 2]) then shows that $\psi(T) \le C$ for all Hilbert-space operators T. Thus there is some interest in determining or estimating the values of C_N.

Crouzeix showed in [5, Theorem 1.1] that $C_2 = 2$. He did this by finding an explicit formula for $\psi(A)$ when $A \in M_2(\mathbb{C})$. He further conjectured that $C_N = 2$ for all $N \ge 2$. This conjecture remains open. There is a substantial amount of supporting numerical evidence (see e.g. [12, 14]), and the conjecture is known to be true for many special classes of matrices (see e.g. [1, 2, 3, 4, 7, 10, 11]). However, even for 3×3 matrices, no universal bound for the Crouzeix ratio seems to be known, beyond the estimate $C_3 \leq 1 + \sqrt{2}$ already mentioned above. The following result, which is our main theorem, may therefore be of interest.

Theorem 1.1. *For each $N \geq 1$, we have the strict inequality $C_N < 1 + \sqrt{2}$.*

In view of the pointwise estimate (5), a natural approach to Theorem 1.1 is to show that the supremum in (6) is always attained. This suggests trying some form of compactness argument. However, a compactness argument presupposes the continuity of the map $A \mapsto \psi(A)$, and it is not hard to see that this map is discontinuous at every multiple of the identity. Worse still, ψ has other discontinuities as well, which turns out to be a more serious obstacle. Fortunately, however, ψ is continuous at a large enough set of points for us to be able to push through the compactness argument, as proposed.

Here, in more detail, is a plan of the article. In §2, we investigate the continuity properties of the map $A \mapsto \psi(A)$ on $M_N(\mathbb{C})$. We show that it is lower semicontinuous everywhere, continuous in some places and discontinuous at others. The proof of Theorem 1.1 is presented in §3. Finally, in §4, we make some concluding remarks and pose some questions.

2. Continuity properties of the Crouzeix ratio

2.1. **Lower semicontinuity.** Our first result shows that ψ is lower semicontinuous on $M_N(\mathbb{C})$.

Theorem 2.1. *If $A_n \to A$ in $M_N(\mathbb{C})$, then*

$$\liminf_{n\to\infty} \psi(A_n) \geq \psi(A). \tag{7}$$

To prove this result, it will be convenient to introduce an auxiliary notion. Given $A \in M_N(\mathbb{C})$ and an open neighbourhood U of $\sigma(A)$, we define the *relative Crouzeix ratio* of the pair (A, U) by

$$\psi_U(A) := \sup\{\|f(A)\| : f \in H^\infty(U),\ |f| \leq 1 \text{ on } U\}.$$

This quantity was considered (under another name) by Crouzeix in [5]. Among other results, he showed in [5, Lemma 2.2] that

$$\psi(A) = \sup_{U \supset W(A)} \psi_U(A), \tag{8}$$

where the supremum is taken over all open neighbourhoods U of $W(A)$.

Proof of Theorem 2.1. Let $A_n \to A$ in $M_N(\mathbb{C})$. Let U be an open neighbourhood of $W(A)$, and let $f \in H^\infty(U)$ with $|f| \leq 1$ on U. For all large enough n, we have $W(A_n) \subset U$, and in particular $\sigma(A_n) \subset U$. For each such n, we clearly have $\|f(A_n)\| \le \psi_U(A_n) \le \psi(A_n)$. Also we have $f(A_n) \to f(A)$ as $n \to \infty$, and in particular $\|f(A_n)\| \to \|f(A)\|$. Hence

$$\liminf_{n\to\infty} \psi(A_n) \ge \liminf_{n\to\infty} \|f(A_n)\| = \|f(A)\|.$$

Taking the supremum of the right-hand side over all $f \in H^\infty(U)$ such that $|f| \le 1$ on U, we deduce that

$$\liminf_{n\to\infty} \psi(A_n) \ge \psi_U(A).$$

Finally, taking the supremum of the right-hand side over all open neighbourhoods U of $W(A)$ and using (8), we obtain the desired conclusion (7). $\square$

2.2. **Upper semicontinuity.** The following result shows that ψ is upper semicontinuous (and hence continuous) at each matrix whose spectrum is contained in the interior of its numerical range.

Theorem 2.2. *Let $A \in M_N(\mathbb{C})$ be such that $\sigma(A) \subset W(A)^\circ$. If $A_n \to A$, then*

$$\limsup_{n\to\infty} \psi(A_n) \le \psi(A).$$

Proof of Theorem 2.2. Suppose for a contradiction, that there exist $\epsilon > 0$ and a sequence (A_n) such that $A_n \to A$ in $M_N(\mathbb{C})$ and

$$\psi(A_n) > \psi(A) + \epsilon \quad (n \ge 1). \tag{9}$$

Then there exist polynomials p_n such that, for all $n \ge 1$,

$$\sup_{W(A_n)} |p_n| \le 1 \quad \text{and} \quad \|p_n(A_n)\| > \psi(A) + \epsilon.$$

Since $A_n \to A$, every compact subset of $W(A)^\circ$ will eventually be contained in $W(A_n)^\circ$. Thus, by a normal-family argument, a subsequence of the p_n (which, by relabelling, we may suppose to be the whole sequence) converges locally uniformly on $W(A)^\circ$ to a holomorphic function f such that $|f| \le 1$ on $W(A)^\circ$. Since $A_n \to A$ and $\sigma(A) \subset W(A)^\circ$, we have $p_n(A_n) \to f(A)$, and in particular $\|p_n(A_n)\| \to \|f(A)\|$. Thus $\|f(A)\| \ge \psi(A) + \epsilon$.

As the whole situation is invariant under translation in the complex plane, there is no loss of generality in supposing, from the outset, that $0 \in W(A)^\circ$. For $r \in (0,1)$, let f_r denote the r-dilation of f, given by $f_r(z) := f(rz)$. Note that $f_r \in A(W(A))$ and that $|f_r| \le 1$ on $W(A)$. Since $f_r \to f$ as $r \to 1^-$ uniformly on a neighbourhood of $\sigma(A)$, it follows that $\|f_r(A)\| \to \|f(A)\|$. In particular, if r is sufficiently close to 1, then $\|f_r(A)\| > \psi(A)$. This contradicts (2). $\square$

2.3. **Discontinuity.** It is easy to see that, if $N \ge 2$, then $A \mapsto \psi(A)$ is discontinuous at $A = 0$. Indeed, since $\psi(\alpha A) = \psi(A)$ for all $\alpha \ne 0$, we have

$$\limsup_{A\to 0} \psi(A) = \sup_{A \in M_N(\mathbb{C})} \psi(A) \ge 2,$$

whereas $\psi(0) = 1$. Since $\psi(A) = \psi(A + \beta I)$ for all $\beta \in \mathbb{C}$, it follows that ψ is discontinuous at every multiple of the identity matrix.

If $N \geq 3$, then ψ is also discontinuous at some matrices A that are not multiples of the identity. The following result gives an example of this sort of discontinuity.

Proposition 2.3. *For $\alpha \geq 0$, let*

$$A_\alpha := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\psi(A_\alpha) \geq \pi/2$ for all $\alpha > 0$ and $\psi(A_0) = 1$. Consequently ψ is discontinuous at A_0.

Proof. Clearly A_0 is self-adjoint, so $\psi(A_0) = 1$.

Now let $\alpha > 0$. Then

$$W(A_\alpha) = \operatorname{conv}\Big(\{z : |z| \leq \alpha/2\} \cup \{1\}\Big),$$

where $\operatorname{conv}(\cdot)$ denotes the convex hull. In particular,

$$W(A_\alpha) \subset S_\alpha := \{z \in \mathbb{C} : |\operatorname{Im} z| \leq \alpha/2\}.$$

Consider the function $f(z) := \tanh(\pi z/(2\alpha))$, which is a conformal mapping of S_α onto the closed unit disk $\overline{\mathbb{D}}$. We have $|f(z)| \leq 1$ for all $z \in W(A_\alpha)$. Also, substituting A_α directly into the Taylor series expansion of $f(z)$, we see that

$$f(A_\alpha) = \begin{pmatrix} f(1) & 0 & 0 \\ 0 & 0 & \alpha f'(0) \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \tanh(\pi/(2\alpha)) & 0 & 0 \\ 0 & 0 & \pi/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular $\|f(A_\alpha)\| \geq \pi/2$. It follows that $\psi(A_\alpha) \geq \pi/2$, as claimed. □

3. Proof of Theorem 1.1

As mentioned in the Introduction, the strategy for proving Theorem 1.1 is to deduce it from the pointwise estimate (5) using a form of compactness argument. This endeavour is complicated by the fact that, as we have just seen, the map $A \mapsto \psi(A)$ is discontinuous at certain points. The discontinuity at multiples of the identity is not a problem, since, using the relation $\psi(A + \beta I) = \psi(A)$, we may consider ψ as being defined on the quotient space $M_N(\mathbb{C})/\mathbb{C}I$, and work on that space. The multiples of the identity then 'disappear'. On the other hand, the discontinuity of ψ at points other than multiples of the identity, such as the one observed in Proposition 2.3, is more problematic. We deal with these by using a decomposition technique to reduce the dimension. The final proof is therefore a combination of a compactness argument and an induction.

The reduction of dimension is the subject of the following theorem. We recall that the constants C_N were defined in (6).

Theorem 3.1. *Let $A \in M_N(\mathbb{C}) \setminus \mathbb{C}I$ be such that $\sigma(A) \cap \partial W(A) \neq \emptyset$. If $A_n \to A$, then*
$$\limsup_{n\to\infty} \psi(A_n) \le C_{N-1}.$$

To prove this theorem, we require two lemmas. The first of these is a decomposition result.

Lemma 3.2. *Let $A \in M_N(\mathbb{C})$. Then A is unitarily equivalent to a block matrix $D \oplus \widetilde{A}$, where D is diagonal, $\sigma(D) \subset \partial W(A)$ and $\sigma(\widetilde{A}) \subset W(A)^\circ$.*

Remark. It is not excluded that D or $\widetilde{A}$ have dimension 0.

Proof. By Schur's theorem, we can suppose that A is lower triangular. The diagonal entries of A are the eigenvalues $\lambda_1, \dots, \lambda_N$. We can suppose that they are ordered so that $\lambda_1, \dots, \lambda_k \in \partial W(A)$ and $\lambda_{k+1}, \dots, \lambda_N \in W(A)^\circ$, where $0 \le k \le N$.

It remains to show that the off-diagonal entries in rows 1 to k are all zero. Let $e_1, \dots, e_N$ be the standard unit vector basis of $\mathbb{C}^N$. Fix j with $1 \le j \le k$ and let $m \in \{j+1, \dots, N\}$. We need to show that $\langle Ae_j, e_m\rangle = 0$. For each $\alpha \in \mathbb{C}$, we have $\|e_j + \alpha e_m\|^2 = 1 + |\alpha|^2$, so, by definition of numerical range,
$$\frac{\big\langle (A(e_j + \alpha e_m),\, (e_j + \alpha e_m)\big\rangle}{1 + |\alpha|^2} \in W(A).$$
Expanding out the left-hand side and using the facts that $\langle Ae_j, e_j\rangle = \lambda_j$ and $\langle Ae_m, e_j\rangle = 0$, we obtain
$$\lambda_j + \overline{\alpha}\langle Ae_j, e_m\rangle + O(|\alpha|^2) \in W(A) \quad (\alpha \in \mathbb{C}, \alpha \to 0).$$
Since $\lambda_j \in \partial W(A)$, this forces $\langle Ae_j, e_m\rangle = 0$, as required. □

The second lemma is a general operator-theory result about the stability of invariant-subspace decompositions.

Lemma 3.3. *Let H be a Hilbert space and let $T \in \mathcal{B}(H)$. Suppose that $H = X \oplus Y$, where X, Y are (not necessarily orthogonal) T-invariant subspaces such that $\sigma(T|_X) \cap \sigma(T|_Y) = \emptyset$. If $T_n \to T$ in $\mathcal{B}(H)$, then there exists a sequence $S_n \to I$ in $\mathcal{B}(H)$ such that, for all sufficiently large n, we have $S_n^{-1}T_nS_n(X) \subset X$ and $S_n^{-1}T_nS_n(Y) \subset Y$.*

Proof. Fix disjoint open subsets U, V of $\mathbb{C}$ such that $\sigma(T|_X) \subset U$ and $\sigma(T|_Y) \subset V$. Then, for all large enough n, we have $\sigma(T_n) \subset U \cup V$. Let P_n, Q_n be the spectral projections of T_n corresponding to U, V respectively. Then $P_n \to P$ and $Q_n \to Q$, where P, Q are projections of H onto X, Y respectively such that $P + Q = I$. Also $P_nQ_n = Q_nP_n = 0$ for all n.

Define $S_n := P_nP + Q_nQ$. Then $S_n \to P^2 + Q^2 = P + Q = I$. In particular, S_n is invertible for all sufficiently large n. For these n, we have $S_n(H) = H$, and so
$$\begin{aligned} P_n(H) &= P_nS_n(H) = P_n(P_nP + Q_nQ)(H) \\ &= P_n^2(X) + P_nQ_n(Y) = P_n(X) = S_n(X). \end{aligned}$$

As $P_n(H)$ is T_n-invariant, we deduce that $T_n S_n(X) \subset S_n(X)$, from which it follows that $S_n^{-1} T_n S_n(X) \subset X$. Likewise $S_n^{-1} T_n S_n(Y) \subset Y$. □

Now we return to Theorem 3.1

Proof of Theorem 3.1. By assumption $A \notin \mathbb{C}I$ and $\sigma(A) \cap \partial W(A) \neq \emptyset$. Using Lemma 3.2, we deduce that A is unitarily equivalent to a block matrix $E \oplus F$ with $E \in M_{N_1}(\mathbb{C})$ and $F \in M_{N_2}(\mathbb{C})$, where $N_1, N_2 \leq N-1$ and $\sigma(E) \cap \sigma(F) = \emptyset$. We can suppose without loss of generality that $A = E \oplus F$.

Let $A_n \to A$ in $M_N(\mathbb{C})$. By Lemma 3.3, applied with $X := \mathbb{C}^{N_1} \oplus 0$ and $Y := 0 \oplus \mathbb{C}^{N_2}$, there exists a sequence $S_n \to I$ in $M_N(\mathbb{C})$ such that, for all sufficiently large n, we have $S_n^{-1} A_n S_n = E_n \oplus F_n$, also a block matrix. Henceforth, we restrict attention to these n.

Let p be a polynomial such that $|p| \leq 1$ on $W(A_n)$. Then

$$\begin{aligned} \|p(A_n)\| &= \|S_n p(E_n \oplus F_n) S_n^{-1}\| \\ &\leq \|S_n\| \|S_n^{-1}\| \|p(E_n \oplus F_n)\| \\ &= \|S_n\| \|S_n^{-1}\| \max\{\|p(E_n)\|, \|p(F_n)\|\} \end{aligned}$$

Now $S_n(X)$ is A_n-invariant and $S_n|_X$ is an invertible map of X onto $S_n(X)$. We further have

$$E_n = (S_n|_X)^{-1} \circ (A_n|_{S_n(X)}) \circ (S_n|_X).$$

Therefore

$$\|p(E_n)\| \leq \|(S_n|_X)^{-1}\| \|p(A_n|_{S_n(X)})\| \|(S_n|_X)\| \leq \|S_n\| \|S_n^{-1}\| \|p(A_n|_{S_n(X)})\|.$$

As $W(A_n|_{S_n(X)}) \subset W(A_n)$, we have $|p| \leq 1$ on $W(A_n|_{S_n(X)})$. Furthermore $\dim S_n(X) = N_1 \leq N-1$. So, by the definition of C_{N_1}, it follows that

$$\|p(A_n|_{S_n(X)})\| \leq C_{N_1} \leq C_{N-1}.$$

We thus obtain that

$$\|p(E_n)\| \leq \|S_n\| \|S_n^{-1}\| C_{N-1}.$$

Likewise for $p(F_n)$. Substituting these estimates into the bound for $\|p(A_n)\|$ above, we obtain

$$\|p(A_n)\| \leq \|S_n\|^2 \|S_n^{-1}\|^2 C_{N-1}.$$

As this holds for all polynomials p with $|p| \leq 1$ on $W(A_n)$, we deduce that

$$\psi(A_n) \leq \|S_n\|^2 \|S_n^{-1}\|^2 C_{N-1}.$$

Letting $n \to \infty$, we obtain

$$\limsup_{n \to \infty} \psi(A_n) \leq \limsup_{n \to \infty} \|S_n\|^2 \|S_n^{-1}\|^2 C_{N-1} = C_{N-1},$$

the last equality because $S_n \to I$. The theorem is proved. □

Finally, we can complete the proof of Theorem 1.1

Proof of Theorem 1.1. The proof is by induction on N. The result is trivial if $N = 1$, since $C_1 = 1$. Suppose now that $N \geq 2$ and that $C_{N-1} < 1 + \sqrt{2}$.

We first claim that the function

$$\widetilde{\psi}(A) := \max\{\psi(A), C_{N-1}\}$$

is upper semicontinuous on $M_N(\mathbb{C}) \setminus \mathbb{C}I$ (and thus continuous there). Indeed, let $A \in M_N(\mathbb{C}) \setminus \mathbb{C}I$, and let $A_n \to A$. If $\sigma(A) \subset W(A)^\circ$, then by Theorem 2.2 we have

$$\limsup_{n\to\infty} \psi(A_n) \leq \psi(A).$$

If, on the other hand, $\sigma(A) \not\subset W(A)^\circ$, then by Theorem 3.1 we have

$$\limsup_{n\to\infty} \psi(A_n) \leq C_{N-1}.$$

Either way, we have

$$\limsup_{n\to\infty} \widetilde{\psi}(A_n) \leq \widetilde{\psi}(A),$$

justifying the claim.

As mentioned at the beginning of this section, the fact that $\psi(A+\beta I) = \psi(A)$ for all $\beta \in \mathbb{C}$ allows us to view ψ as a function defined on the quotient space $M_N(\mathbb{C})/\mathbb{C}I$. The same is therefore true of $\widetilde{\psi}$, and, in addition, $\widetilde{\psi}$ is continuous with respect to the quotient norm, except at 0. As $M_N(\mathbb{C})/\mathbb{C}I$ is finite-dimensional, its unit sphere is compact, so $\widetilde{\psi}$ attains a maximum there, say at A_0.

For $A \in M_N(\mathbb{C})$ and $\alpha \in \mathbb{C} \setminus \{0\}$, we have $\psi(\alpha A) = \psi(A)$, and hence also $\widetilde{\psi}(\alpha A) = \widetilde{\psi}(A)$. Thus in fact $\widetilde{\psi}$ attains a global maximum at A_0, that is,

$$\max\{\psi(A), C_{N-1}\} \leq \max\{\psi(A_0), C_{N-1}\} \quad \forall A \in M_N(\mathbb{C}).$$

Consequently

$$C_N \leq \max\{\psi(A_0), C_{N-1}\}.$$

Finally, we know that $\psi(A_0) < 1 + \sqrt{2}$ by the pointwise estimate (5), and $C_{N-1} < 1 + \sqrt{2}$ by the inductive hypothesis. We may therefore conclude that $C_N < 1 + \sqrt{2}$, thereby completing the induction. $\square$

4. Concluding remarks and questions

(1) Theorem 2.2 can be generalized as follows.

Theorem 4.1. *Let $A \in M_N(\mathbb{C})$. Suppose that $\sigma(A) \cap \partial W(A)$ is either empty or consists exclusively of simple eigenvalues of A. If $A_n \to A$ in $\mathcal{B}(H)$, then*

$$\limsup_{n\to\infty} \psi(A_n) \leq \psi(A).$$

The proof is by combining the techniques used to prove Theorems 2.2 and 3.1. We omit the details, since the result is not needed here. However, it does show that the only 3×3 matrices at which ψ can be discontinuous are those unitarily equivalent to matrices of the form $\alpha A_0 + \beta I$, where A_0 is the matrix in Proposition 2.3. This explains the choice of A_0 in that example.

(2) The constant $\pi/2$ appearing in Proposition 2.3 is not optimal. A more careful analysis shows that, in the notation of Proposition 2.3, we have

$$\psi(A_\alpha) \geq \sup\{|f'(0)| : f \in \operatorname{Hol}(D, \mathbb{D})\} \quad (\alpha > 0),$$

where $\mathbb{D}$ is the open unit disk and $D := \mathbb{D} \cup \{z : \operatorname{Re} z > 0, |\operatorname{Im} z| < 1\}$. By considering $f(1/z)$, we deduce that

$$\psi(A_\alpha) \geq \gamma(K) \quad (\alpha > 0),$$

where K is the compact set formed by taking the complement of the image of D under the map $z \mapsto 1/z$, and $\gamma(K)$ denotes the analytic capacity of K. A simple calculation shows that K is the union of three semi-disks as shown in Figure 1. In particular, since K is connected, its analytic capacity $\gamma(K)$ coincides with its logarithmic capacity $c(K)$. Thus

$$\psi(A_\alpha) \geq c(K) \quad (\alpha > 0).$$

It is plausible that $\limsup_{A \to A_0} \psi(A) = c(K)$, though we cannot prove it. Note, however, that by Theorem 3.1 we must have $\limsup_{A \to A_0} \psi(A) \leq 2$.

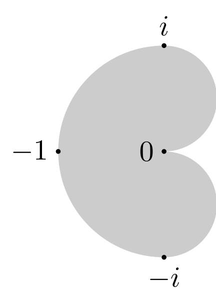

FIGURE 1. The set K

(3) Lemma 3.2 implies a stronger form of itself, in which the inclusion $\sigma(\widetilde{A}) \subset W(A)^\circ$ is replaced by $\sigma(\widetilde{A}) \subset W(\widetilde{A})^\circ$. To see this, it suffices to reapply the lemma with A replaced by $\widetilde{A}$, and repeat as often as necessary.

(4) It would be interesting to quantify the arguments used in proving that $C_N < 1 + \sqrt{2}$ to obtain concrete numerical estimates better than $1 + \sqrt{2}$. Even an estimate for C_3 would be of interest.

(5) Of course, the biggest problem is to identify $C := \lim_{N \to \infty} C_N$. According to Crouzeix's conjecture $C = 2$. Can one at least show that $C < 1 + \sqrt{2}$?

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